In actual use it is observed that (13) converges very rapidly, so that three iterations are sufficient with $N = 30$ for a wide range of dimensions and frequencies. Furthermore, in any of the cases considered the iterated weighting factor Z_N^{I} turned out to be of the order of that Z_{opt} which was found by minimizing $|db_N(Z)/dN|$. For example, in Fig. 2 the values of Z_N^{I} and Z_{opt} differ by a factor of about 3.5.

NUMERICAL RESULTS

The input admittance $y = jb$ of the resonating structure shown in Fig. 1 was computed and is plotted as a function of frequency. Series resonance occurs when the capacitive gap compensates the input impedance of the propagating circular E_{01} mode. The validity of the program has been tested by comparison with Marcuvitz' results for the capacitive gap at low frequencies, which is same as the problem considered here with $\epsilon_r = 1$ [8]. Generation of the orthonormalized functions t_i^{I}, t_k^{II}, and the elements of the matrix G takes about 30 s of CPU time on a CDC-6400 computer for the sum of the modes $N + P = 80$. The CPU time required for the generation and inversion of a 50×50 complex C_N-matrix is about 5 s.

Fig. 2 shows the dependence of the least squares solution b_N on weighting factor Z and mode number N. At the same time an error estimate is provided, since a change of sign exists for the slope of $b_N(Z)$. For comparison, the deviation of $|r_{0N}(Z)|$ from its ideal value $|r_0| = 1$ is also plotted as a measure of power conservation in Fig. 2.

The dashed curve in Fig. 2 refers to susceptances which were computed by minimizing $|db_N(Z)/dN|$ with respect to Z at given values of N. The utilization of the iterated weighting factor Z_N^{I} requires less computer time and results in the dotted curve.

In addition to this, the influence of the upper summation limits N and P on the convergence rate has been studied because the ratio N/P plays an important role for methods which exhibit relative convergence phenomena [9], [10]. In Fig. 3 the dependence of b_N on this ratio is shown as a function of the mode number N. The trend of the minimum value of F_N decreasing with N for a fixed $Z = Z_{opt}$ as seen from Fig. 3 serves as a measure of the convergence rate. It has been found generally that it is sufficient to choose N/P in accordance with point-matching methods, so that in the present case, the near-optimum value of $N/P = R_a/(R_a - R_i)$ lies between 10/5 and 10/6. This coincides with the behavior of F_N shown in Fig. 3 having its maximum average slope F_{10}/F_{50} around this ratio of N/P.

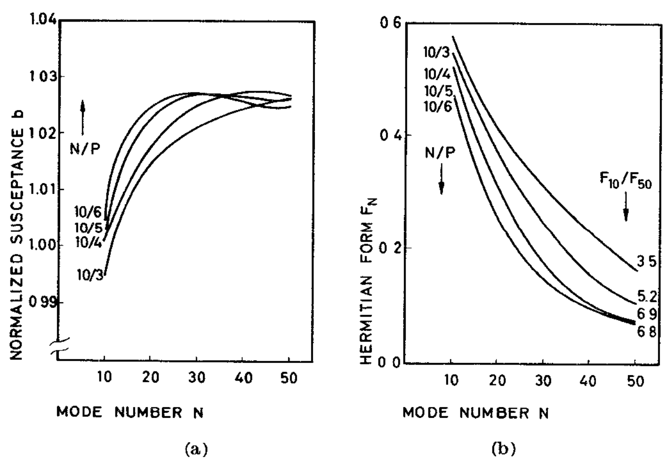

Fig. 3. (a) Influence of the N/P ratio on the convergence of b for near optimum Z. $Z/Z_0 = 8$; $f = 1$ GHz. (b) Corresponding Hermitian forms F_N. The average slope F_{10}/F_{50} serves as a measure of convergence. $f = 1$ GHz.

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The Solution of Electromagnetic Eigenvalue Problems by Least Squares Boundary Residuals

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Abstract—**The least squares boundary residual technique as used for the numerical solution of scattering problems is extended to the solution of electromagnetic eigenvalue problems. The theory is described and numerical results are given for the solution of an L-shaped membrane and microstrip in a hollow conducting guide. The microstrip example was chosen as a test case to compare with Fourier matching. This least square error mimimization technique is of the same family as point matching and Fourier matching; however, it is shown to have three potentially important advantages: 1) it is rigorously convergent, 2) the choice of optimum weighting factors greatly accelerates convergence between a decreasing upper bound and an increasing lower bound, and 3) it is free from problems of relative convergence.**

I. INTRODUCTION

Recently, there has been a surge of interest in the least squares boundary residual technique for the numerical solution of scattering problems [1],[16],[19]. In this short paper, the same approach is extended to the solution of eigenvalue problems, and examples

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are given of solutions of hollow conducting waveguide and of microstrip in a box. An appraisal of the method is given in the final section.

As in the scattering problem, the least squares approach has (in contrast to point-matching [13] and the usual Fourier-matching techniques [3]) the advantage of free parameters that can be chosen to guide convergence between a generally decreasing upper bound and an increasing lower bound. These parameters can be approximated for small order matrices and then used to accelerate convergence with increasing matrix order. Other advantages of the method are (compared with Fourier matching) its freedom from problems of relative convergence [2],[3], and (compared with point matching) its safer criterion, errors being minimized that are global rather than sampled just at discrete points.

To avoid repetition, it will be taken that most of the finer details given for the scattering problem [1] still apply, e.g., choice of basis functions, the use of mathematically convenient interfaces, and aspects of the edge condition. After a brief account of the principles of the method (Section II), results of applications will be given in Section III and a final discussion follows in Section IV.

II. THEORY

In point matching, Fourier matching, and the proposed least squares approach, advantage is taken of the fact that one can easily satisfy the differential equation of the problem. In each method, the region of the problem is divided into a number of subregions, over each of which is taken a truncated series from a complete expansion. In this way, the problem is reduced to approximately satisfying boundary conditions over certain interfaces. These interfaces may be physical ones, such as conducting or dielectric surfaces (boundaries of the original problem) or ones of mathematical convenience that join up the specially introduced subregions.

The boundary residual is defined [1] around the interfaces as a function $R(s)$ that is a linear combination of the total electromagnetic field expansions in adjacent subregions, a combination such that at all points of the interfaces (viz., for all s) the vanishing of $R(s)$ is a necessary condition for the physical solution to the problem. Point matching arranges that these boundary residuals vanish at selected points around the interfaces. Fourier matching (or the method of moments [4]) arranges that $R(s)$ is orthogonal (over the range of s) to a finite number of basis functions. Least squares matching minimizes some weighted integral of $R(s)^2$ over all s. In all cases, the eigenvalues have also to be found that allow the preceding boundary conditions to be best satisfied.

Least square solution of an eigenvalue problem will now be illustrated by consideration of the arbitrarily shaped two-dimensional region in Fig. 1. Suppose it is possible to write expansions of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in each region as

$$\boldsymbol{E}_1 = \sum_{n=1}^{N} a_n \boldsymbol{\phi}_n^{(1)} \qquad \boldsymbol{H}_1 = \sum_{n=1}^{N} a_n \boldsymbol{\psi}_n^{(1)} \tag{1}$$

$$\boldsymbol{E}_2 = \sum_{n=1}^{M} b_n \boldsymbol{\phi}_n^{(2)} \qquad \boldsymbol{H}_2 = \sum_{n=1}^{M} b_n \boldsymbol{\psi}_n^{(2)} \tag{2}$$

which satisfy the boundary conditions on C_1 and C_2, and Maxwell's equations in S_1 and S_2, respectively. The preceding field basis functions $\boldsymbol{\phi}_n^i$ and $\boldsymbol{\psi}_n^i$ must all satisfy the same vector Helmholtz equation

$$(\nabla^2 + k^2)\boldsymbol{v} = 0. \tag{3}$$

The problem now is to obtain the wavenumber k and the coefficients a_n and b_n of (1) and (2) that give the best approximation to $\boldsymbol{E}_1^t = \boldsymbol{E}_2^t$ and $\boldsymbol{H}_1^t = \boldsymbol{H}_2^t$ across the interface C (t denoting the vector component tangential to C). A Hermitian form in a's and b's can now be defined by the functional

$$J(\{a_n\},\{b_n\},k) = \int_C (|\boldsymbol{E}_1^t - \boldsymbol{E}_2^t|^2 + Z^2|\boldsymbol{H}_1^t - \boldsymbol{H}_2^t|^2)\,ds \tag{4}$$

where Z is an arbitrary positive constant with dimensions of impedance. In some problems, only scalar fields will be necessary, rather than the vector form of (1)–(4). Most generally, it is sufficient to consider the components of $\boldsymbol{E}$ and $\boldsymbol{H}$ tangential to the interface C, as described in the scattering problem [1]. In all cases, J is a nonnegative number which can be considered an error norm and which can only be zero for a physically correct eigenvalue k and associated Fourier coefficients $\{a_n\}$ and $\{b_n\}$. The least squares criterion is to require that for given truncated series (1) and (2), the functional $J(\{a_n\}, \{b_n\}, k)$ of (4) be minimized over all possible $\{a_n\}$, $\{b_n\}$, and k. Substituting the expansions (1) and (2) into (4) gives a Hermitian form which can be expressed in matrix form as

$$J(\{a_n\},\{b_n\},k) = \boldsymbol{a}^*A\boldsymbol{a} \tag{5}$$

where the vector $\boldsymbol{a} = (a_1,a_2,\cdots a_N,b_1,b_2,\cdots b_M)$ and A is a Hermitian positive-definite matrix. For a given number of terms, say N and M in (1) and (2), and for a given value of k, the matrix A is determined. To avoid trivial minimization (such as the vanishing of all a's and b's) the minimum is subject to

$$\boldsymbol{a}^*\boldsymbol{a} = 1 \tag{6}$$

or possibly to more complicated normalization [8].

It is known [5] that if (6) is satisfied, then $\boldsymbol{a}^*A\boldsymbol{a}$ of (5) attains its smallest value when $\boldsymbol{a}$ equals the eigenvector associated with the smallest eigenvalue λ_1 of A. The smallest value attained is in fact λ_1, and so by the use of standard computer routines, the minimum of J is found for all possible $\{a_n\}$ and $\{b_n\}$. By computing λ_1 for a range of k values, curves will be obtained of the form shown in Fig. 2, (the problem being solved is described in the following section). In the limit, with $M = N = \infty$, λ_1 would fall to zero at the exact eigenvalues of the problem, at which points (only) the boundary conditions would be satisfied precisely. For any value of k, decreasing M or N will generally raise (and by the minimization definition, cannot lower) the value of λ_1. With practical truncated series, λ_1 will therefore give minimum values near to the correct eigenvalues, such as in Fig. 2. These minima with $N = 5,6,\cdots,10$ give the desired successive approximations to the eigenvalues.

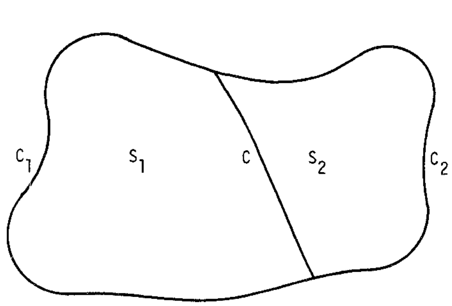

Fig. 1. Arbitrary shaped regions.

III. APPLICATION

Numerical results will now be illustrated with two examples. The first was chosen for its simplicity in introducing the method. The second example is more testing (and complicated) and allows a more "in-depth" comparison of least squares with Fourier matching.

The first example is the L-shaped waveguide or membrane pictured in Fig. 3. This test is simple to implement, without being in any way a "special case." Accurate results of TM modes are also available [6],[7].

By the physical symmetry of the waveguide, all modes can be taken to have a scalar field ϕ that is either odd or even about the symmetry axis A–B of Fig. 3. Considering firstly a TM mode, a scalar field ϕ which satisfies the boundary condition $\phi = 0$ along the conducting walls $y = 0$, $y = \pi$, and $x = 2\pi$, and the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$ inside the structure can be taken as

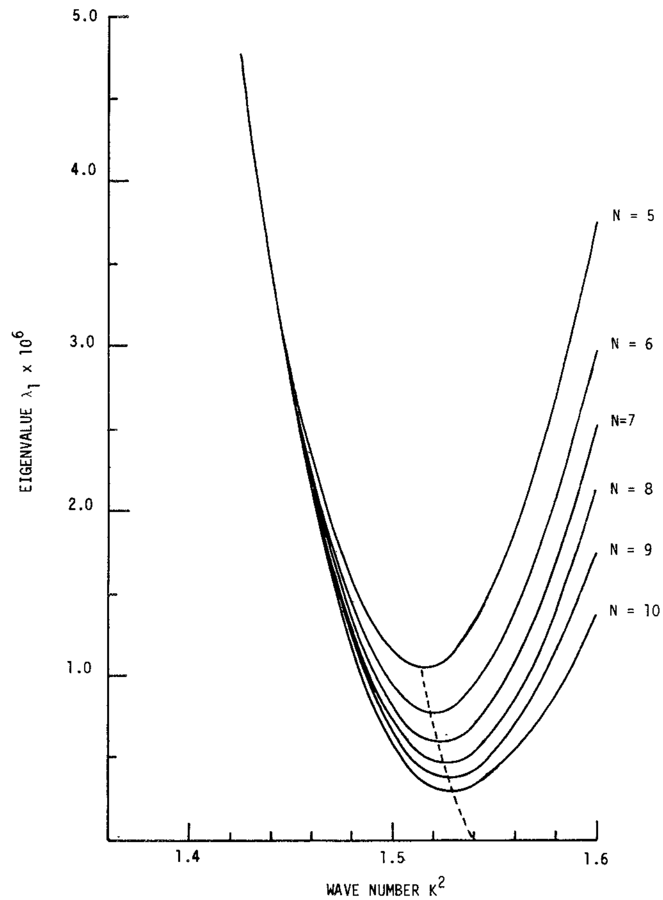

Fig. 2. λ_1 (field error norm) versus k^2 (wavenumber) with increasing number N of expansion terms, for first odd TM modes of Fig. 3. Exact wavenumber k^2 is 1.5398.

$$\phi = \sum_{n=1}^{N} a_n \sin(ny) \sin \gamma_n(x - 2\pi) \tag{7}$$

where $\gamma_n = (k^2 - n^2)^{1/2}$. For odd TM modes, ϕ should also vanish along the symmetry line A–B (viz., $y = x$) and so an appropriate form for (4) is

$$J(\{a_n\},k) = \int_A^B \phi^2 dl = \int_0^{\pi} \sum_{n=1}^{N} a_n \sin(nx) \sin \gamma_n(x - 2\pi) \cdot \sum_{p=1}^{N} a_p \sin(px) \sin \gamma_p(x - 2\pi)\, dx. \tag{8}$$

The elements of matrix A in (5) are therefore

$$A_{np} = \int_0^{\pi} \sin(n\pi) \sin(px) \sin \gamma_n(x - 2\pi) \sin \gamma_p(x - 2\pi)\, dx \tag{9}$$

which can be evaluated easily. This gives A as a Hermitian positive-definite matrix, with all elements either real or imaginary. Multiplying certain rows and columns by j transforms A to a real, symmetric, positive-definite matrix, which is then solved to give the lowest eigenvalue λ_1.

Results are given in Fig. 2 showing curves around a minimum of λ_1, with the matrix order increasing from 5 to 10. If the k values associated with the minimum are plotted against matrix order, a convergence curve is obtained as in Fig. 4.

Curves are given for a variety of "scaling factors." These are factors by which the ϕ defined in (7) has been divided, so that (for instance) the effect of the factor $n \exp(2\pi n)$ is to divide A_{np} of (9) by $np \exp[2\pi(n + p)]$. The fastest convergence in Fig. 4 is clearly for the factor $n \exp(2\pi n)$. The diagonal terms A_{nn} have (for large n) dependence $\exp(4\pi n)$ and so it clearly seems that best

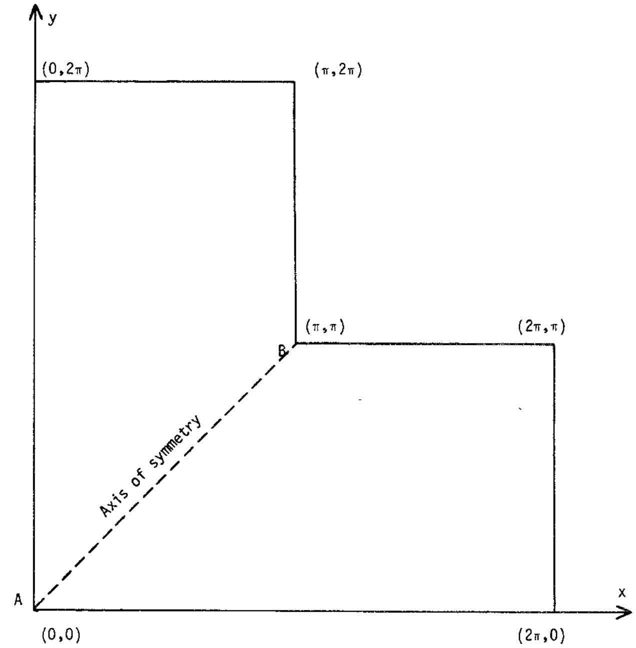

Fig. 3. L-shaped membrane or waveguide.

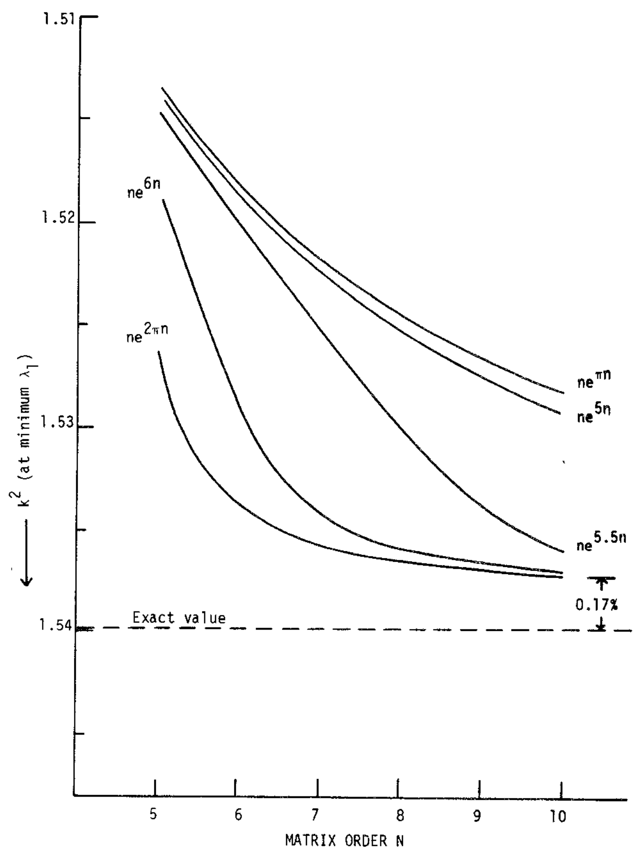

Fig. 4. Effect of scaling factors on convergence with matrix order.

convergence is obtained when scaling eliminates any exponential variation along the diagonal elements of A, or in fact along any row or column. Curves similar to those in Figs. 2 and 4 have been obtained for higher order E modes, including modes that are even or odd about the symmetry line. The procedure for even E modes is identical to that for odd modes, again using (7) as an expansion for ϕ, but minimizing

$$\int_A^B \left(\frac{\partial \phi}{\partial n}\right)^2 dl \quad \text{rather than} \quad \int_A^B \phi^2 dl$$

where $\partial/\partial n$ denotes differentiation normal to A–B. Results are summarized in Table I.

Exact results are obtained for modes 3, 8, and 8a as the correct eigenfunctions are given by either one or two terms from (7). Otherwise errors are generally greater for even modes, as would be expected due to their singular fields near the reentrant corner.

The second and more testing application of the least squares approach is to the eigenvalue problem of microstrip in a hollow conducting guide. Taking advantage of the symmetry plane, just half of the cross section is shown in Fig. 5. Earlier work of the authors [8] on Fourier matching of this structure gave results that suffered from relative convergence [2],[3]; typical results in Fig. 6 show convergence to different limits according to the choice of the ratio of numbers of basis functions in the two regions. This was strictly to be expected as, like similar Fourier approaches [9], no attempt was made to satisfy edge conditions at the microstrip edge. The least squares approach was therefore applied with the expectation [1] of giving more reliable (and perhaps more rapidly convergent) results.

TABLE I
Comparing Computed Values of k^2 (at Minimum λ_1) for E Modes with Published Results [6], [7]

MODE NO.	MODE TYPE	PREVIOUSLY PUBLISHED	LEAST SQUARES
1	Symmetric	0.9767	[a]
2	Antisymmetric	1.5398	1.5393
3	Symmetric square	2.0000	2.0000
4	Antisymmetric	2.9911	2.9913
5	Symmetric	3.2334	3.3271
6	Symmetric	4.2022	4.3023
7	Antisymmetric	4.5542	4.5353
8	Symmetric square	5.0000	5.0000
8a	Antisymmetric	5.0000	5.0000
9	Symmetric	5.7458	5.8383

[a] This result was not produced due to the chance proximity of γ_n to 1, causing loss of significant figures in the evaluation of (9). The problem could be avoided by the use of double-length arithmetic or, better still, by division of each term in (7) by sin $(2\pi\gamma_n)$.

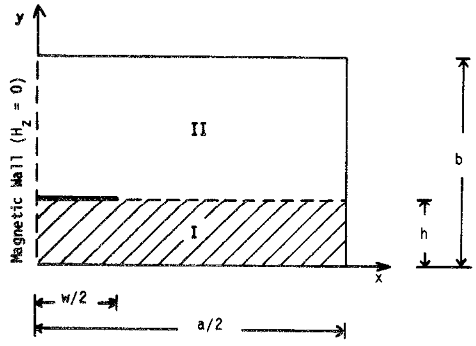

Fig. 5. Half of cross section of microstrip in a conducting box.

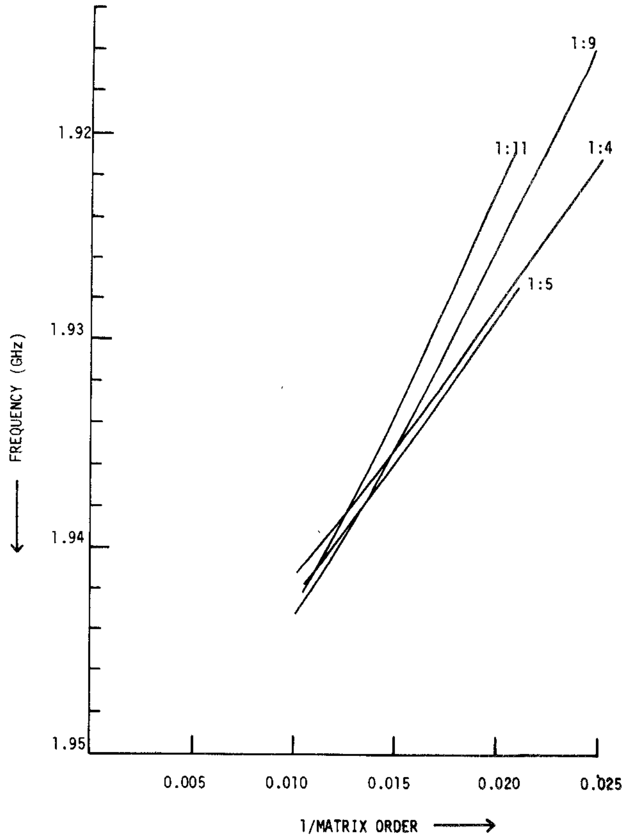

Fig. 6. Frequency versus (1/matrix order) for given β = 100 rad/m, and dimensions $a = b$ = 12.7 mm, $w = h$ = 1.27 mm of Fig. 5. Relative convergence of these "Fourier-matching" results arise from different ratios (Such as 1:11, 1:9, etc.) of numbers of equations in applying boundary conditions to the conducting strip and the air–dielectric interface, respectively.

For both the Fourier matching and least squares approaches, complete expansions are used for the components of $\boldsymbol{E}$ and $\boldsymbol{H}$, with separate expansions in regions I and II of Fig. 5, such as

$$E_z^I = \sum_{n=0}^{N-1} a_n \cos(\alpha_n x) \sin(\gamma_n y) \tag{10}$$

$$H_z^I = (\epsilon_0/\mu_0)^{1/2} \sum_{n=0}^{N-1} b_n \sin(\alpha_n x) \cos(\gamma_n y) \tag{11}$$

where

$$\alpha_n = (2n+1)\pi/a, \quad \gamma_n = (\omega^2\mu\epsilon - \beta^2 - \alpha_n^2)^{1/2}$$

and β is the longitudinal phase propagation constant. Other components of $\boldsymbol{E}$ and $\boldsymbol{H}$ can, of course, be expressed explicitly from (10) and (11). In this way, one derives fields that satisfy Helmholtz equations in both regions I and II, and boundary conditions on the conducting walls $y = 0$, $y = b$, $x = a/2$, and on the magnetic wall $x = 0$. The problem is therefore reduced to arranging that the field expansions also satisfy boundary conditions across the interface at $y = h$, namely, the tangential E fields must vanish along the microstrip $(0 \le x \le w/2)$ and the tangential E and H fields must be continuous across the air–dielectric interface $(w/2 \le x \le a/2)$. Because of the continuity (from region I to II) of tangential E field along the complete boundary $x = 0$ to $x = a/2$, and as identical orthogonal basis functions along $y = h$ are used for $E_{\tan}$ in regions I and II, the equalities $E_z^{I} = E_z^{II}$ and $E_x^{I} = E_x^{II}$ at

$y = h$ can be performed on a term-by-term basis, so giving a simple relation between the individual coefficients a_n and b_n of (10) and (11) and the corresponding coefficients for E^{II} and H^{II}. a_n and b_n are therefore the only "free" field coefficients, and together with ω for given β, they must be chosen to satisfy the boundary conditions along $y = h$ of

$$E_x^{\mathrm{I}} = 0 \text{ and } E_z^{\mathrm{I}} = 0 \text{ along the strip } (0 \leq x \leq w/2) \tag{12}$$

$$H_x^{\mathrm{I}} - H_x^{\mathrm{II}} = 0 \text{ and } H_z^{\mathrm{I}} - H_z^{\mathrm{II}} = 0 \text{ along the dielectric } \cdot (w/2 \leq x \leq a/2). \tag{13}$$

These four boundary conditions of (12) and (13) can be approximated in various ways by Fourier matching. The authors chose [8] to make the field components of [12] orthogonal to M terms from quarter-period sine or cosine expansions over $0 \leq x \leq w/2$, and similarly, the components of (13) orthogonal to N terms from expansions over $w/2 \leq x \leq a/2$. In this way, a comparison was made with earlier results [10],[11], with complete agreement to the accuracy of the published curves. By altering the ratio of numbers of terms (N/M), results were nevertheless found to suffer from relative convergence, as illustrated in Fig. 6. Without special treatment (and then with some delicacy), these problems appear inevitable with Fourier matching [3].

To avoid these problems, possibly to speed up convergence, and to provide a meaningful comparison of least squares with Fourier matching, the same problem was solved by approximating the four boundary conditions of (12) and (13) by a least squares criterion. As outlined in (1), (2), and (4), the four conditions of (12) and (13) can be combined into minimization of the following Hermitian form

$$J(\{a_n\},\{b_n\},\omega) = \int_0^{w/2} \{ |E_x^{\mathrm{I}}|^2 + |E_z^{\mathrm{I}}|^2 \} \, dx + Z^2 \int_{w/2}^{a/2} \{ |H_x^{\mathrm{I}} - H_x^{\mathrm{II}}|^2 + |H_z^{\mathrm{I}} - H_z^{\mathrm{II}}|^2 \} \, dx. \tag{14}$$

By introducing the factor $\gamma_n / |\gamma_n|$ into (10) and (11), the coefficients a_n and b_n can all be taken real, so that (14) is a positive-semidefinite quadratic form in $\{a_n\},\{b_n\}$. Again, as described for (4) and (5), the minimum of (14) is obtained by use of standard matrix routines [12]. In the iris scattering problem [1] an optional electric–magnetic weighting factor was found very effective in speeding up convergence. For this microstrip problem, with four distinct boundary residuals of (12) and (13), three weighting factors A,B,C are available which multiply all E_x, H_x, and H_z values by A, B, and C, respectively, before inclusion in the quadratic form (14). It was found that the weighting factors were best selected by arranging that similar values of contribution came to the quadratic forms from the four boundary residuals of (14). Although naturally more complicated than the scattering case with only one factor, the effects of variation of weighting factors are similar, and typical convergence curves are shown in Fig. 7. The curves show, firstly, the relative convergence of the Fourier approach already referred to above. There is a discrepancy of about 0.5 percent in frequency between results with different N/M ratios. The least squares results can be seen to be, as with the scattering problem [1], generally descending upper bounds or ascending lower bounds. Even more important, the convergence is considerably better for the least squares approach. Comparable accuracy is obtained for a matrix order that is smaller by a factor of about 3, a reduction of about 30 in computing time for the matrix/determinant solution.

Fig. 7 shows convergence to a frequency, resulting in one point of a dispersion characteristic; the total ω–β diagram is given in Fig. 8. Results agree with earlier published curves [10],[11] to their reading accuracy.

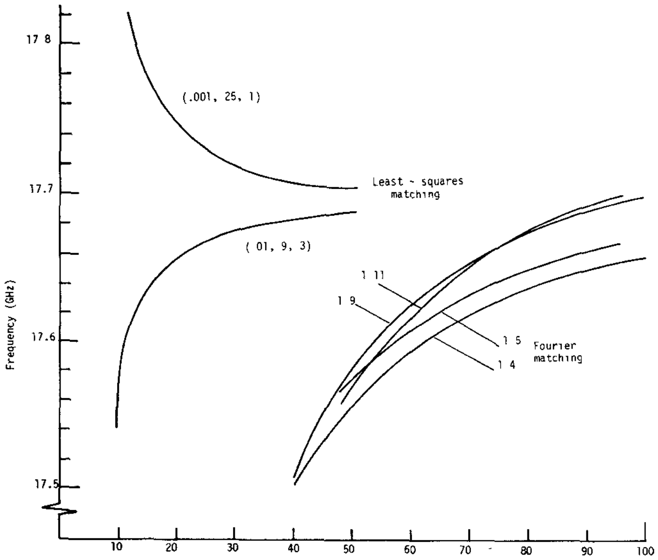

Fig. 7. Convergence curves of frequency versus matrix order for least squares and Fourier matching using the same basis functions. β and dimensions are those of Fig. 6. (A, B, C) are least squares weighting factors referred to in the text. Ratios (such as 1:9) for Fourier matching are described in Fig. 6 and in the text.

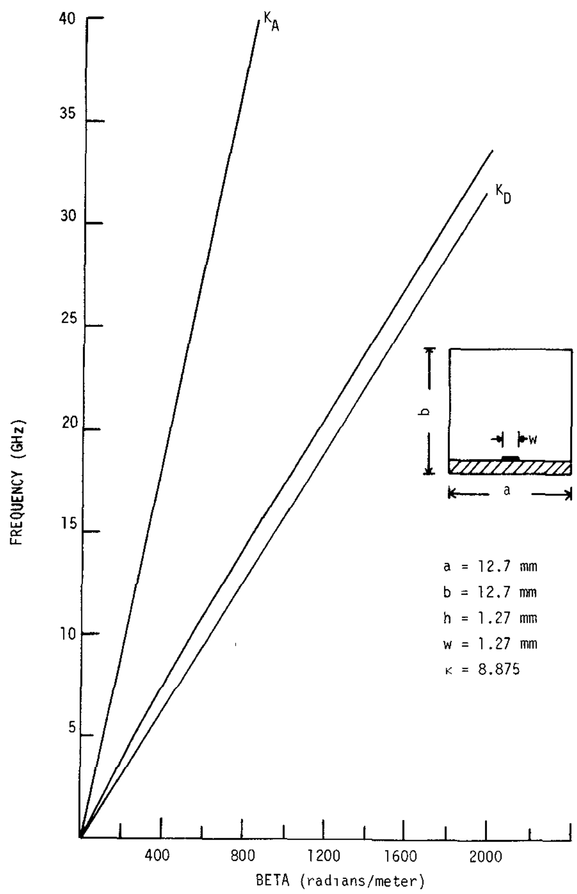

Fig. 8. Resulting dispersion curve for dominant mode of microstrip shown inset (same as in Fig. 6) but via least squares technique.

IV. CONCLUSIONS

As a numerical technique, the least square error minimization is near in style to point matching and Fourier matching. All three methods are versatile in being able to join up a patchwork of regions, each with a complete expansion that exactly satisfies the differential equations of the problem. In each method, some different criterion is put forward to approximately satisfy the remaining boundary conditions.

Point matching is undoubtedly the easiest scheme to implement, as it replaces integration by sampling of fields at discrete points. In its simple form, however, it is known that it can fail to converge or give useful answers [7],[13]. Fourier matching and least squares both involve integration of the boundary residuals (boundary errors). Inner products are needed of all boundary residuals formed with either a set of test functions (Fourier) or the same boundary residuals (least squares).

In the earlier sections, it has been demonstrated that the least squares method, as used for scattering problems, can equally be applied to eigenvalue problems. In Section III, examples are given of two such applications, the microstrip example being taken as a test case to compare with Fourier matching. For this example, the field analysis leading to the formulation of the matrix elements is very similar for the two techniques; in each case, four integrals are required along a boundary interface. The two matrix orders are the same for the same choice of expansion sets. The least squares approach requires the (lowest eigenvalue) solution of a real, symmetric, positive-definite matrix whereas Fourier matching requires the evaluation of the determinant of a real but nonsymmetric and nondefinite matrix. Least squares therefore needs half the storage for the matrix elements. In this work, the least squares matrix was solved by Householder tridiagonalization [12] followed by Sturm sequence and bisection—the Fourier matrix by Gaussian elimination with partial pivoting [14].

For the least squares, solution can be via inverse iteration [15], with Choleski decomposition [14]. Whether using the Fourier or least squares approach, computing time is likely to be comparable for the evaluation of the matrix elements; similarly, for the solution of the matrix. There is therefore little between the methods, in terms of computing time, for a given matrix order.

Overall, the least squares approach would seem to have two potentially important advantages. Firstly, by the empirical choice of optimum weighting factors with low-order matrices these factors can then be used to advantage for higher matrix orders, and considerable acceleration of convergence has been obtained compared with Fourier matching. Best weighting factors were found to occur when arranged (as is easily done) for equal contribution to the error norm from the different boundary residuals. Secondly, in contrast to point matching (and as described in [1],[17], and [18]) it is rigorously convergent.

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The Microstrip Double-Ring Resonator

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***Abstract*—The resonance frequencies and the fields of a microstrip double-ring resonator are discussed. It is shown that no pure even or odd mode can be excited on the resonator. Therefore it is concluded that the microstrip double-ring resonator principally cannot be used to measure the phase velocities of the even and the odd modes on a coupled microstrip line.**

I. INTRODUCTION

Gould and Talboys [1] described a method for measuring the wavelengths on coupled microstrip lines, using a double-ring resonator. The described method has been used by Getsinger [2] too, to prove a theory for calculating the even- and odd-mode wavelengths. Gould and Talboys [1] assumed that an even and an odd mode can be excited on the double-ring resonator despite the fact that the two coupled rings are of different lengths. Furthermore they described that they measured an additional splitting of the resonance frequencies in the case of loosely coupled rings, using a field probe to detect the different resonances.

II. THE STRAIGHT MICROSTRIP DOUBLE-LINE RESONATOR OF DIFFERENT LINE LENGTHS

To get a first insight into the field distribution of a double-ring resonator, the resonator is unrolled and a resonator of two straight coupled microstrip lines, as shown in Fig. 1, is considered. This

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